**Isoparametric Elements**

**Newton-Cotes Example**

Using the Newton-Cotes method with \( i = 2 \) intervals (\( n = 3 \) sampling points), evaluate the integrals:

\[
I = \int_{-1}^{1} x^2 + \cos\left(\frac{x}{2}\right) \, dx \\
I = \int_{-1}^{1} 3^x - x \, dx
\]

\[
I = \int_{-1}^{1} 3^x - x \, dx \approx 2 \left[ \frac{1}{6} (1.3333333) + \frac{4}{6} (1) + \frac{1}{6} (2) \right] \\
= 2.4444444 \\
0.706\% \text{ error}
\]

\[
I = \int_{-1}^{1} 3^x - x \, dx = 2.427305
\]

---

**Isoparametric Elements**

**Gaussian Quadrature**

To evaluate the integral: \( I = \int_{-1}^{1} y \, dx \)

where \( y = y(x) \), we might choose (sample or evaluate) \( y \) at the midpoint \( y(0) = y_1 \) and multiply by the length of the interval, as shown below to arrive at \( I = 2y_1 \), a result that is exact if the curve happens to be a straight line.

---

![Diagram](image-url)
**Isoparametric Elements**

**Gaussian Quadrature**

Generalization of the formula leads to:

\[ I = \int_{-1}^{1} y \, dx = \sum_{j=1}^{n} W_j y(x_j) \]

That is, to approximate the integral, we evaluate the function at several sampling points \( n \), multiply each value \( y_j \) by the appropriate weight \( W_j \), and add the terms.

Gauss's method chooses the sampling points so that for a given number of points, the best possible accuracy is obtained.

Sampling points are located symmetrically with respect to the center of the interval.

**Isoparametric Elements**

**Gaussian Quadrature**

Generalization of the formula leads to:

\[ I = \int_{-1}^{1} y \, dx = \sum_{j=1}^{n} W_j y(x_j) \]

In general, Gaussian quadrature using \( n \) points (Gauss points) is exact if the integrand is a polynomial of degree \( 2n - 1 \) or less.

In using \( n \) points, we effectively replace the given function \( y = f(x) \) by a polynomial of degree \( 2n - 1 \).

The accuracy of the numerical integration depends on how well the polynomial fits the given curve.
Isoparametric Elements

Gaussian Quadrature

Generalization of the formula leads to:

$$ I = \int_{-1}^{1} y \, dx = \sum_{i=1}^{n} W_i y(x_i) $$

If the function $f(x)$ is not a polynomial, Gaussian quadrature is inexact, but it becomes more accurate as more Gauss points are used.

Also, it is important to understand that the ratio of two polynomials is, in general, not a polynomial; therefore, Gaussian quadrature will not yield exact integration of the ratio.

Isoparametric Elements

Gaussian Quadrature - Two-Point Formula

To illustrate the derivation of a two-point ($n = 2$) consider:

$$ I = \int_{-1}^{1} y \, dx = W_1 y(x_1) + W_2 y(x_2) $$

There are four unknown parameters to determine: $W_1, W_2, x_1,$ and $x_2$.

Therefore, we assume a cubic function for $y$ as follows:

$$ y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 $$
Isoparametric Elements

Gaussian Quadrature - Two-Point Formula

In general, with four parameters in the two-point formula, we would expect the Gauss formula to exactly predict the area under the curve.

\[ A = \int_{-1}^{1} \left( C_0 + C_1 x + C_2 x^2 + C_3 x^3 \right) dx = 2C_0 + \frac{2}{3}C_2 \]

However, we will assume, based on Gauss's method, that \( W_1 = W_2 \) and that \( x_1 = x_2 \) as we use two symmetrically located Gauss points at \( x = \pm a \) with equal weights.

The area predicted by Gauss's formula is

\[ A_G = W_1 y(-a) + W_2 y(a) = 2W \left( C_0 + C_2 a^2 \right) \]

Isoparametric Elements

Gaussian Quadrature - Two-Point Formula

If the error, \( e = A - A_G \), is to vanish for any \( C_0 \) and \( C_2 \), we must have, in the error expression:

\[ \frac{\partial e}{\partial C_0} = 0 = 2 - 2W \quad \Rightarrow \quad W = 1 \]

\[ \frac{\partial e}{\partial C_2} = 0 = \frac{2}{3} - 2a^2W \quad \Rightarrow \quad a = \frac{1}{\sqrt{3}} = 0.5773.... \]

Now \( W = 1 \) and \( a = 0.5773 \) ... are the \( W_i \)'s and \( a_i \)'s (\( x_i \)'s) for the two-point Gaussian quadrature as given in the table.
Isoparametric Elements

Gaussian Quadrature Example

Use three-point Gaussian Quadrature evaluate the integrals:

\[ I = \int_{-1}^{1} \left[ x^2 + \cos \left( \frac{x}{2} \right) \right] dx \]

\[ I \approx \sum_{i=1}^{3} W_i \left[ x_i^2 + \cos \left( \frac{x_i}{2} \right) \right] \]

\[ \approx \frac{5}{9} (1.5259328) + \frac{8}{9} (1.0) + \frac{5}{9} (1.5259328) = 2.5843698 \]

0.00004% error

\[ I = \int_{-1}^{1} \left[ 3^x - x \right] dx \]

\[ \approx \sum_{i=1}^{3} W_i \left[ 3^{x_i} - x_i \right] \]

\[ \approx \frac{5}{9} (1.2015923) + \frac{8}{9} (1.0) + \frac{5}{9} (1.5673475) = 2.4271888 \]

0.00477% error
Isoparametric Elements

Gaussian Quadrature Example

In two dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other as

\[ I = \int_{-1}^{1} \int_{-1}^{1} f(s,t) \, ds \, dt \approx \int_{-1}^{1} \left[ \sum_{i=1}^{n} W_i f(s_i,t) \right] \, dt \]

\[ \approx \sum_{j=1}^{n} W_j \left[ \sum_{i=1}^{n} W_i f(s_i,t_j) \right] \]

\[ \approx \sum_{i=1}^{n} \sum_{j=1}^{n} W_i W_j f(s_i,t_j) \]

Isoparametric Elements

Gaussian Quadrature Example

For example, a four-point Gauss rule (often described as a 2 x 2 rule) is shown below with \( i = 1, 2 \) and \( j = 1, 2 \) yields

\[ I \approx \sum_{i=1}^{2} \sum_{j=1}^{2} W_i W_j f(s_i,t_j) \approx W_1 W_1 f(s_1,t_1) + W_1 W_2 f(s_1,t_2) + W_2 W_1 f(s_2,t_1) + W_2 W_2 f(s_2,t_2) \]

The four sampling points are at \( s_i \) and \( t_i = \pm 0.5773... \) and \( W_i = 1.0 \)
Isoparametric Elements

Gaussian Quadrature Example

In three dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other two as

\[
I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(s, t, z) ds \, dt \, dz \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} W_i W_j W_k f(s_i, t_j, z_k)
\]

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

For the two-dimensional element, we have shown in previous chapters that

\[
[k] = \int_A [B]^T [D][B] h \, dx \, dy
\]

where, in general, the integrand is a function of \(x\) and \(y\) and nodal coordinate values.
**Isoparametric Elements**

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

We have shown that \([k]\) for a quadrilateral element can be evaluated in terms of a local set of coordinates \(s-t\), with limits from -1 to 1 within the element.

\[
[k] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] h \left| J \right| ds \, dt
\]

Each coefficient of the integrand \([B]^T [D][B] \left| J \right|\) evaluated by numerical integration in the same manner as \(f(s, t)\) was integrated.

---

**Isoparametric Elements**

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

A flowchart to evaluate \([k]\) for an element using four-point Gaussian quadrature is shown here.
**Isoparametric Elements**

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

The explicit form for four-point Gaussian quadrature (now using the single summation notation with $i = 1, 2, 3, 4$), we have

$$
[k] = \int_1^1 \int_1^1 [B]^T [D][B] h [J] ds \, dt
$$

$$
= \left[ B(s_1, t_1) \right]^T [D] \left[ B(s_1, t_1) \right] \left[ J(s_1, t_1) \right] W_1 W_1
+ \left[ B(s_2, t_2) \right]^T [D] \left[ B(s_2, t_2) \right] \left[ J(s_2, t_2) \right] W_2 W_2
+ \left[ B(s_3, t_3) \right]^T [D] \left[ B(s_3, t_3) \right] \left[ J(s_3, t_3) \right] W_3 W_3
+ \left[ B(s_4, t_4) \right]^T [D] \left[ B(s_4, t_4) \right] \left[ J(s_4, t_4) \right] W_4 W_4
$$

where $s_1 = t_1 = -0.5773$, $s_2 = -0.5773$, $t_2 = 0.5773$, $s_3 = 0.5773$, $t_3 = -0.5773$, and $s_4 = t_4 = 0.5773$ and $W_1 = W_2 = W_3 = W_4 = 1.0$

---

**Isoparametric Elements**

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

Evaluate the stiffness matrix for the quadrilateral element shown below using the four-point Gaussian quadrature rule.

Let $E = 30 \times 10^6$ psi and $\nu = 0.25$. The global coordinates are shown in inches. Assume $h = 1$ in.
**Isoparametric Elements**

Evaluation of the Stiffness Matrix by Gaussian Quadrature

Using the four-point rule, the four points are:

\[(s_1, t_1) = (-0.5773, -0.5773)\]
\[(s_2, t_2) = (-0.5773, 0.5773)\]
\[(s_3, t_3) = (0.5773, -0.5773)\]
\[(s_4, t_4) = (0.5773, 0.5773)\]

With \(W_1 = W_2 = W_3 = W_4 = 1.0\)

\[
\begin{bmatrix}
B(s_1, t_1) & D & B(s_1, t_1) \\
B(s_2, t_2) & D & B(s_2, t_2) \\
B(s_3, t_3) & D & B(s_3, t_3) \\
B(s_4, t_4) & D & B(s_4, t_4)
\end{bmatrix}
\]

\[
\text{First evaluate } \left|J\right| \text{ at each Gauss, for example:}
\]

\[
\left|J(-0.5773, -0.5773)\right|
\]
**Isoparametric Elements**

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

Recall:

\[
\left[ J \right] = \frac{1}{8} \left( \{ X_c \}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{ Y_c \} \right)
\]

\[
\{ X_c \}^T = [x_1, x_2, x_3, x_4] \quad \{ Y_c \}^T = [y_1, y_2, y_3, y_4]
\]

For this example:

\[
\{ X_c \}^T = [3, 5, 5, 3] \quad \{ Y_c \}^T = [2, 2, 4, 4]
\]

**Isoparametric Elements**

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

Recall:

\[
\left[ J(0.5773, -0.5773) \right]
\]

\[
\left[ J(-0.5773, 0.5773) \right]
\]

\[
\left[ J(-0.5773, -0.5773) \right]
\]

\[
\left[ J(0.5773, -0.5773) \right]
\]

\[
\left[ J(0.5773, 0.5773) \right]
\]

\[
= 1.000
\]

Similarly:

\[
\left[ J(-0.5773, 0.5773) \right] = 1.000
\]

\[
\left[ J(0.5773, -0.5773) \right] = 1.000
\]

\[
\left[ J(0.5773, 0.5773) \right] = 1.000
\]
Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

The shape functions are computed as:

\[ N_{ts} = \frac{1}{4}(t - 1) - \frac{1}{4}((-0.5773) - 1) = -0.3943 \]

\[ N_{t} = \frac{1}{4}(t - 1) - \frac{1}{4}((-0.5773) - 1) = -0.3943 \]

Similarly, \([B_2]\), \([B_3]\), and \([B_4]\) must be evaluated like \([B_1]\) at (-0.5773, -0.5773).

We then repeat the calculations to evaluate \([B]\) at the other Gauss points.

\[
\begin{bmatrix}
-0.1057 & 0 & 0.1057 & 0 & 0 & -0.1057 & 0 & -0.3943 \\
-0.1057 & -0.1057 & -0.3943 & 0.1057 & 0.3943 & 0 & -0.3943 & 0 \\
0 & 0.3943 & 0 & 0.1057 & 0.3943 & 0.3943 & -0.1057 & -0.3943 \\
\end{bmatrix}
\]

With similar expressions for \([B(-0.5773, 0.5773)]\), and so on.
**Isoparametric Elements**

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

The matrix $[D]$ is:

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \times 10^6 \text{ psi}$$

Finally, $[k]$ is:


**Axisymmetric Elements**

**Problems**

25. To be assigned from your textbook “A First Course in the Finite Element Method” by D. Logan.
End of Chapter 10