Linear Programming: Model Formulation and Graphical Solution

Chapter Topics

- Model Formulation
- A Maximization Model Example
- Graphical Solutions of Linear Programming Models
- A Minimization Model Example
- Irregular Types of Linear Programming Models
- Characteristics of Linear Programming Problems
Objectives of business decisions frequently involve maximizing profit or minimizing costs.

Linear programming is an analytical technique in which linear algebraic relationships represent a firm's decisions, given a business objective, and resource constraints.

Steps in application:
- Identify problem as solvable by linear programming.
- Formulate a mathematical model of the unstructured problem.
- Solve the model.
- Implementation

Model Components

- **Decision variables** - mathematical symbols representing levels of activity of a firm.
- **Objective function** - a linear mathematical relationship describing an objective of the firm, in terms of decision variables - this function is to be maximized or minimized.
- **Constraints** – requirements or restrictions placed on the firm by the operating environment, stated in linear relationships of the decision variables.
- **Parameters** - numerical coefficients and constants used in the objective function and constraints.
Summary of Model Formulation Steps

**Step 1**: Clearly define the decision variables

**Step 2**: Construct the objective function

**Step 3**: Formulate the constraints

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**LP Model Formulation: A Maximization Example (1)**

- **Product mix problem - Beaver Creek Pottery Company**
- **How many bowls and mugs should be produced to maximize profits given labor and materials constraints?**
- **Product resource requirements and unit profit:**

<table>
<thead>
<tr>
<th>Product</th>
<th>Resource Requirements</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Labor (hr/unit) Clay (lb/unit) Profit ($/unit)</td>
<td></td>
</tr>
<tr>
<td>Bowl</td>
<td>1 4</td>
<td>40</td>
</tr>
<tr>
<td>Mug</td>
<td>2 3</td>
<td>50</td>
</tr>
</tbody>
</table>
LP Model Formulation: A Maximization Example (2)

- **Resource**: 40 hrs of labor per day
- **Availability**: 120 lbs of clay
- **Decision Variables**: 
  - $x_1$ = number of bowls to produce per day
  - $x_2$ = number of mugs to produce per day
- **Objective Function**: Maximize $Z = 40x_1 + 50x_2$
- **Resource Constraints**: 
  - $x_1 + 2x_2 \leq 40$ hours of labor
  - $4x_1 + 3x_2 \leq 120$ pounds of clay
- **Non-Negativity Constraints**: $x_1 \geq 0; x_2 \geq 0$
LP Model Formulation: A Maximization Example (3)

Complete Linear Programming Model:

Maximize   \[ Z = 40x_1 + 50x_2 \]
subject to: \[ \begin{align*}
1x_1 + 2x_2 & \leq 40 \\
4x_1 + 3x_2 & \leq 120 \\
x_1, x_2 & \geq 0
\end{align*} \]

A feasible solution does not violate any of the constraints:
Example  \[ x_1 = 5 \text{ bowls} \]
\[ x_2 = 10 \text{ mugs} \]
\[ Z = 40x_1 + 50x_2 = 700 \]

Labor constraint check:
\[ 1(5) + 2(10) = 25 < 40 \text{ hours, within constraint} \]

Clay constraint check:
\[ 4(5) + 3(10) = 50 < 120 \text{ pounds, within constraint} \]
Infeasible Solutions

- An **infeasible solution** violates at least one of the constraints:
  
  Example $x_1 = 10$ bowls  
  $x_2 = 20$ mugs  
  $Z = $1400

  Labor constraint check:  
  $1(10) + 2(20) = 50 > 40$ hours, violates constraint

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Graphical Solution of LP Models

- Graphical solution is limited to linear programming models containing only two decision variables (can be used with three variables but only with great difficulty).

- Graphical methods provide visualization of how a solution for a linear programming problem is obtained.
Coordinate Axes

Maximize $Z = 40x_1 + 50x_2$
subject to: 
1. $x_1 + 2x_2 \leq 40$
2. $4x_1 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$

Labor Constraint

Maximize $Z = 40x_1 + 50x_2$
subject to: 
1. $x_1 + 2x_2 \leq 40$
2. $4x_1 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$
Labor Constraint Area

Maximize $Z = 40x_1 + 50x_2$
subject to:  
\[ x_1 + 2x_2 \leq 40 \]
\[ 4x_1 + 3x_2 \leq 120 \]
\[ x_1, x_2 \geq 0 \]

Clay Constraint Area

Maximize $Z = 40x_1 + 50x_2$
subject to:  
\[ x_1 + 2x_2 \leq 40 \]
\[ 4x_1 + 3x_2 \leq 120 \]
\[ x_1, x_2 \geq 0 \]
Both Constraints

Maximize $Z = 40x_1 + 50x_2$
subject to:
1. $x_1 + 2x_2 \leq 40$
2. $4x_1 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$

Feasible Solution Area

Maximize $Z = 40x_1 + 50x_2$
subject to:
1. $x_1 + 2x_2 \leq 40$
2. $4x_1 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$
Objective Function Solution = $800

Maximize $Z = 40x_1 + 50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

Alternative Objective Function Solution Lines

Maximize $Z = 40x_1 + 50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$
Optimal Solution

Maximize $Z = 40x_1 + 50x_2$
subject to:
1. $x_1 + 2x_2 \leq 40$
2. $4x_1 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$

Optimal Solution Coordinates

Maximize $Z = 40x_1 + 50x_2$
subject to:
1. $x_1 + 2x_2 \leq 40$
2. $4x_1 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$
**Extreme (Corner) Point Solutions**

Maximize \( Z = 40x_1 + 50x_2 \)
subject to:  
\[
1x_1 + 2x_2 \leq 40 \\
4x_1 + 3x_2 \leq 120 \\
x_1, x_2 \geq 0
\]

**Optimal Solution for New Objective Function**

Maximize \( Z = 70x_1 + 20x_2 \)
subject to:  
\[
1x_1 + 2x_2 \leq 40 \\
4x_1 + 3x_2 \leq 120 \\
x_1, x_2 \geq 0
\]
Slack Variables

- Standard form requires that all constraints be in the form of equations (equalities).
- A slack variable is added to a \( \leq \) constraint (weak inequality) to convert it to an equation (=).
- A slack variable typically represents an unused resource.
- A slack variable contributes nothing to the objective function value.

Linear Programming Model: Standard Form

Max \( Z = 40x_1 + 50x_2 \)
subject to:
\[
\begin{align*}
1x_1 + 2x_2 + s_1 &= 40 \\
4x_1 + 3x_2 + s_2 &= 120 \\
x_1, x_2, s_1, s_2 &\geq 0
\end{align*}
\]

Where:
- \( x_1 \) = number of bowls
- \( x_2 \) = number of mugs
- \( s_1, s_2 \) are slack variables
LP Model Formulation: A Minimization Example

- Two brands of fertilizer available - Super-Gro, Crop-Quick.
- Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
- Super-Gro costs $6 per bag, Crop-Quick $3 per bag.
- Problem: How much of each brand to purchase to minimize total cost of fertilizer given following data?

<table>
<thead>
<tr>
<th>Brand</th>
<th>Nitrogen (lb/bag)</th>
<th>Phosphate (lb/bag)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Super-gro</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Crop-quick</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 2.15: Fertilizing Farmer’s field
LP Model Formulation: A Minimization Example

Decision Variables:
\[ x_1 = \text{bags of Super-Gro} \]
\[ x_2 = \text{bags of Crop-Quick} \]

The Objective Function:
Minimize \[ Z = 6x_1 + 3x_2 \]
Where: \[ 6x_1 = \text{cost of bags of Super-Gro} \]
\[ 3x_2 = \text{cost of bags of Crop-Quick} \]

Model Constraints:
\[ 2x_1 + 4x_2 \geq 16 \text{ lb (nitrogen constraint)} \]
\[ 4x_1 + 3x_2 \geq 24 \text{ lb (phosphate constraint)} \]
\[ x_1, x_2 \geq 0 \text{ (non-negativity constraint)} \]
Feasible Solution Area: A Minimization Example

Minimize \( Z = 6x_1 + 3x_2 \)
subject to:
\[
\begin{align*}
2x_1 + 4x_2 & \geq 16 \\
4x_1 + 3x_2 & \geq 24 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Optimal Solution Point: A Minimization Example

Minimize \( Z = 6x_1 + 3x_2 \)
subject to:
\[
\begin{align*}
2x_1 + 4x_2 & \geq 16 \\
4x_1 + 3x_2 & \geq 24 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Surplus Variables: A Minimization Example

- A surplus variable is subtracted from a \(\geq\) constraint to convert it to an equation (=).
- A surplus variable represents an excess above a constraint requirement level.
- Surplus variables contribute nothing to the calculated value of the objective function.
- Subtracting slack variables in the farmer problem constraints:

\[
2x_1 + 4x_2 - s_1 = 16 \text{ (nitrogen)} \\
4x_1 + 3x_2 - s_2 = 24 \text{ (phosphate)}
\]

Graphical Solutions: A Minimization Example

Minimize \( Z = 6x_1 + 3x_2 + 0s_1 + 0s_2 \)
subject to:

\[
2x_1 + 4x_2 - s_1 = 16 \\
4x_1 + 3x_2 - s_2 = 24 \\
x_1, x_2, s_1, s_2 \geq 0
\]
Irregular Types of Linear Programming Problems

- For some linear programming models, the general rules do not apply.
- Special types of problems include those with:
  - Multiple optimal solutions
  - Infeasible solutions
  - Unbounded solutions

Multiple Optimal Solutions: Beaver Creek Pottery Example

Objective function is parallel to a constraint line.

Maximize $Z = 40x_1 + 30x_2$
subject to: $x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

Where:
$x_1 = \text{number of bowls}$
$x_2 = \text{number of mugs}$

Point $B$
$x_1 = 24$
$x_2 = 8$
$Z = 1,200$

Point $C$
$x_1 = 30$
$x_2 = 0$
$Z = 1,200$
An Infeasible Problem

Every possible solution violates at least one constraint:
Maximize \( Z = 5x_1 + 3x_2 \)
subject to: \( 4x_1 + 2x_2 \leq 8 \)
\( x_1 \geq 4 \)
\( x_2 \geq 6 \)
\( x_1, x_2 \geq 0 \)

An Unbounded Problem

Value of objective function increases indefinitely:
Maximize \( Z = 4x_1 + 2x_2 \)
subject to: \( x_1 \geq 4 \)
\( x_2 \leq 2 \)
\( x_1, x_2 \geq 0 \)
Characteristics of Linear Programming Problems

- A linear programming problem requires a decision - a choice amongst alternative courses of action.
- The decision is represented in the model by decision variables.
- The problem encompasses a goal, expressed as an objective function, that the decision maker wants to achieve.
- Constraints exist that limit the extent of achievement of the objective.
- The objective and constraints must be definable by linear mathematical functional relationships.

Properties of Linear Programming Models

- **Proportionality** - The rate of change (slope) of the objective function and constraint equations is constant.
- **Additivity** - Terms in the objective function and constraint equations must be additive.
- **Divisibility** - Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.
- **Certainty** - Values of all the model parameters are assumed to be known with certainty (non-probabilistic).
Problem Statement

Example Problem No. 1 (1 of 3)

- Hot dog mixture in 1000-pound batches.
- Two ingredients, chicken ($3/lb) and beef ($5/lb).
- Recipe requirements:
  - at least 500 pounds of chicken
  - at least 200 pounds of beef
- Ratio of chicken to beef must be at least 2 to 1.
- Determine optimal mixture of ingredients that will minimize costs.

Solution: Example Problem No. 1 (2 of 3)

Step 1:
Identify decision variables.

\[ x_1 = \text{lb of chicken in mixture (1000 lb.)} \]
\[ x_2 = \text{lb of beef in mixture (1000 lb.)} \]

Step 2:
Formulate the objective function.

Minimize \[ Z = 3x_1 + 5x_2 \]
where \( Z \) = cost per 1,000-lb batch
\( 3x_1 \) = cost of chicken
\( 5x_2 \) = cost of beef
Solution: Example Problem No. 1 (3 of 3)

Step 3: Establish Model Constraints

\[ x_1 + x_2 = 1,000 \text{ lb} \]
\[ x_1 \geq 500 \text{ lb of chicken} \]
\[ x_2 \geq 200 \text{ lb of beef} \]
\[ \frac{x_1}{x_2} \geq \frac{2}{1} \text{ or } x_1 - 2x_2 \geq 0 \]
\[ x_1, x_2 \geq 0 \]

The Model: Minimize \( Z = 3x_1 + 5x_2 \)
subject to: \( x_1 + x_2 = 1,000 \text{ lb} \)
\[ x_1 \geq 50 \]
\[ x_2 \geq 200 \]
\[ x_1 - 2x_2 \geq 0 \]
\[ x_1, x_2 \geq 0 \]

Example Problem No. 2 (1 of 3)

Solve the following model graphically:

Maximize \( Z = 4x_1 + 5x_2 \)
subject to: \( x_1 + 2x_2 \leq 10 \)
\[ 6x_1 + 6x_2 \leq 36 \]
\[ x_1 \leq 4 \]
\[ x_1, x_2 \geq 0 \]

Step 1: Plot the constraints as equations
Example Problem No. 2 (2 of 3)

Maximize $Z = 4x_1 + 5x_2$
subject to:  \[ x_1 + 2x_2 \leq 10 \]
\[ 6x_1 + 6x_2 \leq 36 \]
\[ x_1 \leq 4 \]
\[ x_1, x_2 \geq 0 \]

Step 2: Determine the feasible solution space

Example Problem No. 2 (3 of 3)

Maximize $Z = 4x_1 + 5x_2$
subject to:  \[ x_1 + 2x_2 \leq 10 \]
\[ 6x_1 + 6x_2 \leq 36 \]
\[ x_1 \leq 4 \]
\[ x_1, x_2 \geq 0 \]

Step 3 and 4: Determine the solution points and optimal solution